Models of Objective Chance - An Analysis through Examples

Gergei Bana

Abstract

In his seminal work, A Subjectivist's Guide to Objective Chance, David Lewis considered the possibility for a subjectivist to accept the existence of objective chance, and argued how this acceptance would affect the subjectivist's degrees of belief (credences) about the world: they have to satisfy the Principal Principle. Lewis did not put his proposal into mathematically precise terms. Most importantly, he did not define what kind of mathematical object objective chance was. In this work we pay careful attention to identify what mathematical model the subjectivist agent considers, and through several simple examples we illustrate how objective chance can be incorporated in the subjectivist's model in a mathematically rigorous manner.

1 Introduction

In A Subjectivist's Guide to Objective Chance, David Lewis (1980) considered the possibility for a subjectivist to accept the existence of objective chance, and argued how this acceptance would affect the subjectivist's degrees of belief (credences) about the world. He asked what constraints the acceptance of the existence of objective chance imposed on the credences that a reasonable subjectivist could assign to sets of possible worlds. The starting point of Lewis (1980) is that we accept the existence of a time dependent chance ch_t, and that both objective chance and credence should satisfy the usual rules of probability theory. Lewis's answer to the above question is that the only limitation on the credence function (measure) is that for any chancy event A, conditioned on the evidence the agent has collected and on that the objective chance of A is known to be r, the credence of A also has to equal r:

$$C(A|ch_t(A) = r \wedge E) = r$$

as long as E is "admissible", that is, as long as the evidence collected does not force us to have a degree of belief other than r.

Lewis did not state his proposal in a mathematically rigorous manner. In the second part of his work, Lewis (1980) mentions that objective chance is a chance distribution $P_{tw}(A)$ that assigns a probability to event A, and it can depend on time t and on possible world w but does not investigates what properties this function should satisfy. To this day a lot of research is being done on the Principal Principle, especially around the mystery of what admissibility means. These works seem to us rather speculative as long as we do not specify mathematically clearly what we are talking about.

At the PSA Biennial Meeting in 2014, there were some attempts to treat at least some particular questions about the Principal Principle in a mathematically precise manner by Rédei and Gyenis (2016) and Bana (2016). While preparing the latter, I realized that a clear understanding of what objective chance is mathematically is entirely missing from the literature.

My aim here is to look at several specific examples, through which we can identify what kind of mathematical object corresponds to objective chance in its most general sense. I believe that the mathematical model we give is the most general one in which the notion of objective chance can be meaningful. We do not rely on special assumptions for our analysis, I try to be completely faithful to Lewis's original 1980 work, but make it more explicit.

In this work I do not attempt to treat the Principal Principle itself. The sole purpose here is to motivate a rigorous definition of objective chance, and I shall work out the consequences of this definition on the Principal Principle elsewhere.

Furthermore, in this work I do not attempt to formalize Lewis's later views (for example, Lewis (1994)) on how objective chance is given by a best system. We simply propose a general modeling of objective chance that can accommodate any kind of objective chances, let it be propensity, relative frequencies or determined by a best system.

2 The Agent and His Model

We shall be careful throughout this work detaching the model of reality from reality itself. We do not follow Lewis in speculating what objective chance in reality is, as it does not matter. What matters is what the agent's view about reality is, what model of possible worlds the agent choses as a starting point in which he places the pieces of evidence he collects. *As Lewis's starting point is the assumption that objective chance is real, our main task is to see how to incorporate objective chance rigorously in the mathematical model of possible worlds.* We want to give a general mathematical definition that objective chance must minimally satisfy let it be a propensity or reflect relative frequencies.

Lewis did not make explicit how the subjectivist agent views reality, and how he collects evidence, so let us make this explicit here:

- **Subjecivist Agent's View:** The subjectivist agent has a set W, such that each $w \in W$ represents a possible world. In the agent's opinion, each $w \in W$ describes facts that are either true or not in the real world. The agent assumes that in this W there is a w_{real} corresponding to the real world, but he does not know which w it is. So, he assigns degrees of belief, C called credence to subsets $S \subseteq W$, and $C(S)$ represents his degree of belief that $w_{\mathrm{real}} \in S$. Lewis assumes that C satisfies the laws of probability. Accordingly, on W a σ-algebra Σ of subsets is defied, and C is a probability distribution on Σ. We can call (W, Σ) the *subjectivist's model* of the world. The agent, one way or other, we do not care how, collects evidence. What we do care about is that whenever he collects a piece of evidence, he places that in his mathematical model, that is, he assigns a set $E \in \Sigma$ to it. Having collected such a piece of evidence E, from that on the agent assumes that $w_{\text{real}} \in E$, meaning that he updates his credence to $C'(\,\cdot\,) := C(\,\cdot\,|E)$. When the agent learns one piece of evidence after the other, $E_1, E_2, ..., E_n \in \Sigma$, then he updates his credence to $C_n(\,\cdot\,) := C(\,\cdot\,|E_1 \wedge E_2 \wedge .... \wedge E_n)$.

The set of possible worlds, W, may be produced many ways. It may be a set of possible worlds that abide a certain theory of physics, it can be about coin toss, etc. One way or other, we do not care, it has to be defined mathematically precisely for the subjectivist agent to assign his credences.

E could be a set of W the elements of which are possible worlds corresponding to manufacturing a coin in a certain way, or to forming a certain radioactive material, or to a certain prediction of a crystal ball, etc. Normally, this happens the following way: An observable, such as the outcome of a coin toss, a radioactive decay, etc. is represented mathematically as a random variable (measurable function) X on W. To each possible world $w \in W$, a value $X(w)$ of the observable belongs. Measuring the value of the random variable, and obtaining say x, the set corresponding to this value, $E = \{w|X(w) = x\}$ is in Σ (as X is measurable), and this is an example of an evidence.

Note, this all is the agent's view! *It is entirely irrelevant for us whether the agent's model is a good model of the world or not. We only care about that he has a model and that pieces of evidence are placed in this model.*

Having accepted this modeling of the subjectivist agent's view, next question is how to accommodate objective chance in this model in a mathematically rigorous manner. For the subjectivist who believes in objective chance, objective chance has to be part of the model of possible worlds, hence it should be defined somehow on W. Namely, when the objective chance of event A, that is, $ch_t(A)$ can have different possible values in the agent's view, that means that each of these different possible values of $ch_t(A)$ must signify a different possible world. In other words, to each possible world w, a single value $ch_{tw}(A)$ must belong. Let us rather denote it by $ch_t(w, A)$. Can the function $(w, A) \mapsto ch_t(w, A)$ be an arbitrary one, or does it have to satisfy certain properties? Clearly, as objective chance is a probability, for fixed t and w, the function $A \mapsto ch_t(w, A)$ should be a probability measure over some algebra, as Lewis required it. The main objective of this work is to investigate whether we need to require some other properties as well for ch_t that Lewis did not notice. We shall see that a chancy situation comes with some structures on the set of possible worlds that Lewis failed to consider, and the above function is not arbitrary relative to these structures. A further question is what happens when we allow time to change as well, what restrictions do we get for the function $t \mapsto ch_t$. More generally, what are the requirements that would make a family of objective chances $\{ch_i\}_{i\in I}$ consistent? This is a non trivial question, as it is easy to give examples of contradictory objective chances. However, this question is out of the scope of this paper, we shall only focus on what makes a function $(w, A) \mapsto ch(w, A)$ an objective chance function.

Through the examples below, I attempt to make a convincing case for the mathematical definition I propose in Section 4.

3 Examples

In this section I provide numerous examples to support my proposition for modeling objective chance. First we look at examples in which objective chance is a propensity, while in the last example objective chance corresponds to relative frequencies.

3.1 The Basics

Our first example is straight out of Lewis (1980): "... suppose you are not sure that the coin is fair. You divide your belief among three alternative hypotheses about the chance of heads, as follows.

- You believe to degree 27% that the chance of heads is 50%.
- You believe to degree 22% that the chance of heads is 35%.
- You believe to degree 51% that the chance of heads is 80%.

Then to what degree should you believe that the coin falls heads? Answer. (27% x 50%)+(22% x 35%)+(51% x 80%); that is, 62%."

3.1.1 A Set of Possible Worlds

How can we treat this in a mathematically rigorous manner? Perhaps the simplest set of possible worlds that can accommodate this experiment is when first a coin is forged with three kinds of possible biasses, and then the coin is tossed with heads or tails as possible outcomes. Perhaps the simplest imaginable mathematical model for this is the following: The set of possible worlds have elements of the form of ordered pairs (x, y) where x is one of the symbols b_1, b_2 or b_2 corresponding to the three possible biasses, and y is either symbol h or symbol t corresponding to heads and tails:

$$W := \left\{ (x, y) \;\middle|\; (x = b_1 \vee x = b_2 \vee x = b_3) \wedge (y = h \vee y = t) \right\}.$$

Accordingly, the world in which the proposition "the coin is manufactured so that it has the second kind of bias and the outcome of the coin toss is heads" is satisfied corresponds to the mathematical object $(b_2, h) \in W$. Let B_i (for $i = 1, 2, 3$) be the proposition that "the coin was manufactured with the i'th kind of bias". The subset of W corresponding to this proposition is denoted by $[B_i]$, and is

$$[B_i] := \{(b_i, h), (b_i, t)\} \subset W.$$

The set corresponding to the proposition (denoted by H) that "the outcome of the coin toss is heads" is

$$[H] := \{(b_1, h), (b_2, h), (b_3, h)\} \subset W,$$

and similarly for tails, $[T]$.

We can also think of $[H]$ and $[T]$ the following way: the outcomes of the coin toss is in fact a random variable on W:

$$V : W \to \{h, t\}, \qquad V(\,(x, y)\,) = y.$$

Then

$$[H] = V^{-1}(\{h\}) \qquad \text{and} \qquad [T] = V^{-1}(\{t\}).$$

3.1.2 The Event Algebra of Chancy Outcomes and Objective Chance

We still have to tell what the chance *ch* of $[H]$ and of $[T]$ are. How can we incorporate chance as a mathematical object in the model we have defined? Note that there is a sub-algebra[1] Σ_{chancy} of Σ corresponding to the outcomes of the coin toss generated by $[H]$ and $[T]$:

$$\Sigma_{\text{chancy}} = \{\ \emptyset\ ,\ [H]\ ,\ [T]\ ,\ W\ \}$$

In other words, Σ_{chancy} is the algebra we obtain by pulling back the discrete measure of $\{h, t\}$ with V.

According to Lewis's example, if the coin was manufactured the first way, then the chance of heads is 0.5, if the second way, then the chance of heads is 0.35, and if the third way, then the chance of heads is 0.8. That is, there is no single $ch([H])$, it has three possible values. In other words, *ch* cannot be a measure on Σ_{chancy}. However, for each possible $w \in W$, we can define $ch(w, [H])$. And this is completely intuitive: On each possible world, $[H]$ has some objective chance, and it of course can vary from possible world to possible world. In fact, Lewis thought of this too: $ch(w, A)$ is nothing but his *chance distribution* on page 276 of Lewis (1980).

We have the following:

$$ch((x, y), [H]) := \begin{cases} 0.5 \text{ if } x = b_1 \\ 0.35 \text{ if } x = b_2 \\ 0.8 \text{ if } x = b_3 \end{cases}$$

Then, setting

$$ch((x, y), [T]) := 1 - ch((x, y), [H]), \quad ch((x, y), \emptyset) := 0, \quad ch((x, y), W) := 1$$

for each $w \in W$, the function

$$ch(w, \cdot) : A \mapsto ch(w, A)$$

becomes a probability measure on Σ_{chancy}, corresponding to the fact that on each possible world, the objective chance is a probability distribution on the possible outcomes.

[1] That is, closed under finite union and complementation

Note that for all $A \in \Sigma_{\text{chancy}}$, the function

$$\mathit{ch}(\cdot, A) : w \mapsto \mathit{ch}(w, A)$$

is constant on the sets $[B_1]$, $[B_2]$, and $[B_3]$. This corresponds to the fact that the objective chance depends only on how the coin was manufactured. In other words, $\mathit{ch}(\cdot, A)$ *depends only on the past* before the coin is tossed. The set corresponding to the proposition $\mathit{ch}(\cdot, A) = r$, notated as $[\mathit{ch}(\cdot, A) = r]$ is $[B_1]$ if $r = 0.5$, $[B_2]$ if $r = 0.35$, and $[B_3]$ if $r = 0.8$, otherwise it is the emptyset.

3.1.3 The Event Algebra of the Probabilistic Setup and Historic Evidence

What is *historic evidence* in our example at the point when the coin is tossed? Clearly, statements about how the coin was manufactured, what kind of setup was carried out before the coin is tossed. Hence mathematically, in this example, historic evidence is an element in the subalgebra of events $\Sigma_1 \subset \Sigma$ generated by $[B_1]$, $[B_2]$, and $[B_3]$ as atoms. Written explicitly,

$$\Sigma_1 := \{ \emptyset , [B_1] , [B_2] , [B_3] , [B_1] \cup [B_2] , [B_2] \cup [B_3] , [B_1] \cup [B_3] , W \}.$$

For those who are familiar theory of stochastic processes, setting $\Sigma_0 := \{ \emptyset \}$ and $\Sigma_2 := \Sigma$, the sequence $\Sigma_0, \Sigma_1, \Sigma_2$ is a *filtration*: an assignment of increasing event algebras to time steps. The index i is time, it means "step i": 0 before the process starts, 1 for the forging of the coin and 2 two for the coin toss. Functions that are measurable[2] with respect to Σ_i are those that can be computed based on the information up until the i'th step is carried out. For example, a function that is measurable with respect to Σ_1 can depend on x, but not on y in $(x, y) \in W$ as by step 1 the coin has been forged, but not tossed.

We introduce the notation Σ_{env}, which shall stand for the chancy environment, in this example

$$\Sigma_{\text{env}} := \Sigma_1$$

meaning that the chancy events are considered in the environment when the past until step 1 is fixed.

The property that $\mathit{ch}(\cdot, A)$ is constant on the sets $[B_1]$, $[B_2]$, and $[B_3]$ (the atoms of Σ_{env}) can be equivalently reformulated as $\mathit{ch}(\cdot, A)$ is measurable with respect to $\Sigma_{\text{env}} = \Sigma_1$. As a consequence, an explicit hypothesis about the objective chance, $[\mathit{ch}(\cdot, A) \in I]$, where $I \subseteq [0, 1]$ an interval, is in Σ_1, hence giving such an explicit hypothesis is a special way of giving historical information in this example.

Note that for example,

$$\mathit{ch}((b_2, h), [H]) = \mathit{ch}((b_2, t), [H]) = 0.35,$$

[2]Recall that a function $f : W \to \mathbb{R}$ is measurable with respect to Σ σ-algebra on W if and only if for all intervals $I \subseteq \mathbb{R}$, the set $f^{-1}(I) := \{w \in W : f(w) \in I\} \in \Sigma$. In case Σ is atomic (finite spaces are always such), measurable functions are those that are constant on the atoms of Σ.

even though $(b_2, t) \notin [H]$. This corresponds to the fact that although on (b_2, t), once the coin was manufactured to have b_2 bias, the chance of heads is 0.35, but it does not actually come about, because on this possible world the coin lands tails.

3.1.4 Extending the Chancy Algebra

If desired, for each w, the chance of any event in Σ can be defined relative to Σ_{env}. Let $\overline{ch}$ denote the extended chance. Now $\overline{ch}(\cdot, S)$ should be measurable with respect to Σ_{env} for all $S \in \Sigma$ corresponding to the idea that we are considering objective chance once the coin was manufactured with some bias. We can do this the following way. First let us ask the question: How could we define, say, $\overline{ch}((b_1, t), [B_2])$? Clearly, once the coin was manufactured to have bias b_1, that is to be fair, it can land heads or tails with probability 0.5, but it cannot change its bias. So $\overline{ch}((b_1, t), [B_2]) = 0$. For the same reason, $\overline{ch}((b_1, t), [B_1]) = 1$. In general, we set for all $E \in \Sigma_{\text{env}}$,

$$\overline{ch}(w, E) := \begin{cases} 1 \text{ if } w \in E \\ 0 \text{ if } w \notin E \end{cases}$$

and then for any $A \in \Sigma_{\text{chancy}}$ and $E \in \Sigma_{\text{env}}$,

$$\overline{ch}(w, E \cap A) := \overline{ch}(w, E) \cdot ch(w, A).$$

As all elements in Σ can be written as the union of sets of the form $E \cap A$ as above (the event algebra Σ is generated by Σ_{env} and Σ_{chancy}), this defines a unique probability measure $S \mapsto \overline{ch}(w, S)$ on Σ, and the function $w \mapsto \overline{ch}(w, S)$ is Σ_{env}-measurable for all $S \in \Sigma$.

3.1.5 Observations on the Structure

To summarize, we have seen in this example that to model the chancy situation, we can define a subalgebra Σ_{chancy} representing the chancy events of outcomes, and another subalgebra Σ_{env} representing the events about how the chancy experiment is set up, that is, how the coin is forged. Furthermore, we had a chance assignment $ch : W \times \Sigma_{\text{chancy}} \to [0, 1]$ such that for all $w \in W$, the function $ch(w, \cdot)$ is a probability measure on Σ_{chancy}, and for each $A \in \Sigma_{\text{chancy}}$, the function $ch(\cdot, A)$ is measurable with respect to Σ_{env}. These structures, Σ_{env} and Σ_{chancy} are both missing from Lewis's analysis and from the followup literature as well, but I think they are essential aspects of objective chance, and no rigorous analysis is possible without them.

3.2 What Is Time?

Let us continue by making our first example more complex. Suppose that the coin forging process is the following: It is first decided in some unknown way whether the coin to be forged shall be fair or not. If it is not going to be fair, then it is decided in a fair manner—for example, by throwing an old fair coin—whether the new coin shall be forged such that the bias allows a chance of 0.35 for heads, or a chance of 0.8. Once the coin is forged, it is thrown 3 times.

What is the new W? For example, the following. Similarly to the first example, let the symbol b_1 stand for the forging of the new coin in the first, fair way. Let a stand for forging the new coin biassed. Let b_2 and b_3 denote the forging of the coin the second and the third way again. Let t_1, t_2, t_3 and h_1, h_2, h_3 stand for the outcomes of the coin tosses. Clearly, forging the biassed coins is a longer process because it involves an extra decision. Let

$$W := W_1 \cup W_2$$

where

$$W_1 := [B_1] := \left\{ (x, y_1, y_2, y_3) \,\middle|\, x = b_1,\ (y_i = t_i) \vee (y_i = h_i) \right\}$$

and

$$W_2 := \left\{ (w, x, y_1, y_2, y_3) \,\middle|\, w = a,\ (x = b_2 \vee x = b_3),\ (y_i = t_i) \vee (y_i = h_i) \right\}$$

Let π_i denote the projection on the i'th entry. For example, $\pi_2(\,(w, x, y_1, y_2, y_3)\,) = x$ and $\pi_2(\,(x, y_1, y_2, y_3)\,) = y_1$. For a set $S \in W$, let

$$\pi_i(S) := \{\ z \mid \text{there is a } w \in W \text{ such that } z = \pi_i(w)\ \}.$$

We take again $\Sigma := \mathcal{P}(W)$, that is, all subsets of W. Here too we have a filtration: $\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$ where $\Sigma_0 := \emptyset$, $\Sigma_5 := \Sigma$, and for $j = 1, 2, 3, 4$, Σ_j is generated by the sets of the form $S \in \Sigma$ such that $\pi_1(S)$, ..., $\pi_j(S)$ each has a single element, and S contains all elements of W that start with $\pi_1(S)$, ..., $\pi_j(S)$:

$$\Sigma_j = \overline{\left\{ S \in \Sigma \,\middle|\, \begin{array}{l} \text{each of } \pi_1(S), \ldots, \pi_j(S) \text{ contains exactly one element,} \\ \text{and for all } w \in W \text{, if } w \text{ is such that} \\ \pi_1(w) \in \pi_1(S)\ \wedge\ \ldots\ \wedge\ \pi_j(w) \in \pi_j(S) \text{, then } w \in S \end{array} \right\}}$$

where the overline means closure under making unions and complementation. That is, Σ_j is generated by sets S such that all the elements of each S have the same history until the j'th step and their future after j is completely undetermined.

For example, the set $\{(b_1, t_1, t_2, t_3), (b_1, t_1, t_2, h_3)\} \in \Sigma_3$ is such a set where the history until the third step is determined to be b_1, t_1, t_2, and after the third step it is completely open (t_3 or h_3). Or, $\{(a, b_2, t_1, h_2, h_3), (a, b_2, t_1, h_2, t_3), (a, b_2, t_1, t_2, h_3), (a, b_2, t_1, t_2, t_3)\} \in \Sigma_3$ is also such a set, where the history until the third step is fixed to be a, b_2, t_1, while the last two steps are completely open and can be h_2, h_3, or t_2, h_3, or h_2, t_3, or t_2, t_3. On the other hand, $\{(a, b_2, t_1, t_2, h_3), (a, b_2, t_1, h_2, t_3)\}$ is not such a set and is not in Σ_3, because although the history until the third step is fixed to be a, b, t_1, the last two are not completely undetermined as only t_2, h_3 and h_2, t_3 appear in the set. A function

$f : W \to \mathbb{R}$ that has different values on $\{(a, b_2, t_1, t_2, h_3), (a, b_2, t_1, h_2, t_3)\} \subset W$ and on $\{(a, b_2, t_1, h_2, h_3), (a, b_2, t_1, t_2, t_3)\} \subset W$ is not determined by the first three entries a, b_2, t_1 only, but is also dependent on the last two.

Suppose we want to consider the objective chance that the second coin lands heads or tails. Then we set

$$\Sigma_{\text{chancy}} = \{ \emptyset , [H_2] , [T_2] , W \}$$

where H_2 (or T_2) denotes the proposition that the outcome of the second coin toss is heads (or tails), and where now $[H_2] = [H_2]_1 \cup [H_2]_2$ with

$$[H_2]_1 = \left\{ (x, y_1, y_2, y_3) \in W \;\middle|\; x = b_1,\ y_2 = h_2 \right\}$$

and

$$[H_2]_2 = \left\{ (w, x, y_1, y_2, y_3) \in W \;\middle|\; w = a,\ y_2 = h_2 \right\}$$

and similarly for $[T_2]$.

Suppose we are concerned about the objective chance of the second coin toss of the new coin. But at what time? It cannot of course be considered at step 0, because it is not known how it is decided whether the new coin shall be biassed or not. However, it could be considered at step 1, after the first decision was made, because the rest relies on coin tosses with objective chance. Or, another natural time is once the new coin is forged, and before it is used. However, on W_1 this happens at step 1, while on W_2 this toss happens at step 2. Or, we could also consider it at the time just before the coin is thrown for the second time.

3.2.1 After the First Step

In this case we set the time to $\tau := 1$. This means that it has been decided whether a fair or a biassed coin is to be forged, but nothing else. Clearly, the algebra representing the environment of the experiment is

$$\Sigma_{\text{env}} = \Sigma_1 = \{ \emptyset , W_1 , W_2 , W \}$$

As on W_2, the bias might be two kinds, each with probability $1/2$, the chance distribution is

$$ch_1(w, [H_2]) = \begin{cases} 0.5 \text{ if } w \in W_1 \\ 0.5 \cdot 0.35 + 0.5 \cdot 0.8 = 0.575 \text{ if } w \in W_2 \end{cases}$$

And $ch_1(w, [T_2]) = 1 - ch_1(w, [H_2])$.

Exactly the same way as we did in Section 3.1.4, here too we can extend chance onto the algebra generated by Σ_{env} and Σ_{chancy}, denoted by $\overline{\Sigma_{\text{env}} \cup \Sigma_{\text{chancy}}}$, but here this set is not the entire Σ.

Since after the first step all outcomes are assumed to be chancy, namely, the decision of the bias via a fair coin and then the coin toss three times, we can actually define objective chance for all those events. In this case we can consider the objective chance distribution on the event algebra:

$$\Sigma_{\text{chancy}} := \overline{\{\emptyset\ ,\ [B_2]\ ,\ [B_3]\ ,\ [H_1]\ ,\ [T_1]\ ,\ [H_2]\ ,\ [T_2]\ ,\ [H_3]\ ,\ [T_3]\ ,\ W\}}$$

where

$$[B_2] := \left\{ (w, x, y_1, y_2, y_3) \,\middle|\, w = a, x = b_2,\ (y_i = t_i) \vee (y_i = h_i) \right\}$$

$$[B_3] := \left\{ (w, x, y_1, y_2, y_3) \,\middle|\, w = a, x = b_3,\ (y_i = t_i) \vee (y_i = h_i) \right\}$$

Note that $[B_1] = W \setminus ([B_2] \cup [B_3])$, so $[B_1] \in \Sigma_{\text{chancy}}$. In fact, Σ_{chancy} in this case is just the full Σ. Clearly, $ch_1(w, [B_1]) = 1$, $ch_2(w, [B_2]) = ch_1(w, [B_3]) = 0$ if $w \in W_1$, $ch_1(w, [B_1]) = 0$, $ch_2(w, [B_2]) = ch_1(w, [B_3]) = 0.5$ if $w \in W_2$, $ch_1(w, [B_1] \cap [H_1] \cap [H_2] \cap [H_3]) = 0.5^3$ if $w \in W_1$, and so on.

3.2.2 At Some Stopping Time

It also make sense to consider objective chance after the coin is forged. However, this happens at different times on different possible worlds. So consider the following random variable:

$$\tau(w) = \begin{cases} 1 \text{ on } W_1 \\ 2 \text{ on } W_2 \end{cases}$$

Such a function is called *stopping time* in stochastic processes. Namely, given a filtration $(\Sigma_n)_{n\in\mathbb{N}}$ on a probability space W, a random variable $\tau : W \to \mathbb{N}$ is a stopping time, if for all $n \in \mathbb{N}$, the set where τ is less than or equal to n, $[\tau \leq n] = \{w \in W : \tau(w) \leq n\}$ is in Σ_n. Such a stopping time determines an event algebra:

$$\Sigma_\tau := \overline{\bigcup_{n\in\mathbb{N}} \{\ E \mid E \in \Sigma_n \wedge E \subseteq [\tau \leq n]\ \}}$$

In our case,

$$\Sigma_\tau = \{\ \emptyset\ ,\ [B_1]\ ,\ [B_2]\ ,\ [B_3]\ ,\ [B_1] \cup [B_2]\ ,\ [B_2] \cup [B_3]\ ,\ [B_1] \cup [B_3]\ ,\ W\ \}$$

(that is, $W_1 = [B_1]$ is not split, but W_2 is split into $[B_2]$ and $[B_3]$). Let now $\Sigma_{\text{env}} := \Sigma_\tau$. Again, if we take Σ_{chancy} to be

$$\Sigma_{\text{chancy}} := \{\ \emptyset\ ,\ [H_2]\ ,\ [T_2]\ ,\ W\ \},$$

then the same way as before, on W_1:

$$ch_\tau((b_1, y_1, y_2, y_3), [H_2]) := 0.5$$

and

$$ch_\tau((a, x, y_1, y_2, y_3), [H_2]) := \begin{cases} 0.35 \text{ if } x = b_2 \\ 0.8 \text{ if } x = b_3 \end{cases}$$

It is easy to see that ch_τ is measurable with respect to $\Sigma_{\text{env}} = \Sigma_\tau$.

3.2.3 Observations on the Structure

In this section, we looked at a case where objective chance comes from a stochastic process (but in which not all branchings have associated probabilities). In case of a stochastic process, for each time t, there is an associated event algebra Σ_t representing the events that happened until time t. As more events happen until a later time, $\Sigma_{t_1} \subseteq \Sigma_{t_2}$ for $t_1 < t_2$. When we consider an event at a certain time determined by the past, that certain time might vary from possible world to possible world. Let τ be such function assigning to each possible world w a time $\tau(w)$. As τ is determined by the past, it is necessary that for each time value t, the set $[\tau \leq t] := \{w \in W | \tau(w) \leq t\}$ is in Σ_t. Such a τ is called stopping time. Σ_τ contains all events that happen until τ. In case of objective chance ch_τ (of chancy events Σ_{chancy}) determined by the past until τ, the environmental algebra Σ_{env} equals Σ_τ. Accordingly, for each $A \in \Sigma_{\text{chancy}}$, the function $ch_\tau(\cdot, A)$ must be measurable with respect to $\Sigma_{\text{env}} = \Sigma_\tau$. And of course, for each $w \in W$, the function $ch_\tau(w, \cdot)$ is a probability measure over Σ_{chancy}. Again, the notion of filtration and stopping time are missing from the literature on the Principal Principle and objective chance, but they are really essential for understanding time-dependent objective chance.

3.3 Is It Always the Past?

In the previous examples, the environmental algebra Σ_{env} always agreed with a Σ_τ algebra representing the events of the past with respect to a stopping time τ. This however does not have to be so. In fact, even Lewis (1994) later considered objective chance that depended on the future. What matters is how we set up our problem: what is given, and what is left chancy. It makes perfect sense to talk about the objective chance of a coin toss in the past, having fixed some conditions at a later time. It makes perfect sense to consider objective chance relative to a Σ_{env} other than Σ_τ for some stopping time τ. Consider again the example in Section 3.2.2. Having forged the coin, knowing whether it was forged biassed or not, and already seeing the result of the first toss, the agent knowing the process but not knowing which bias the coin has in his world, he can consider the objective chance of not only the second coin toss, but also the objective chance of having forged the first, second or the third type of coin. Why not? Consider the forging process described in Section 3.2.2. Given that the coin was forged biassed and the outcome of the first coin toss is heads, nothing prevents us to compute the objective chance of having a second type coin or a third type coin, and the objective chances of the outcomes of the next coin toss. Given that the coin was forged fair and that the outcome of the first toss is heads, we again have no problem telling the chance of having a third-type coin: it is 0.

Accordingly, we fix Σ_{env} the following way:

$$\Sigma_{\text{env}} := \overline{\{\ \emptyset\ ,\ [H_1^f]\ ,\ [T_1^f]\ ,\ [H_1^b]\ ,\ [T_1^b]\ ,\ W\}}$$

where the index stands for fair and biassed. $[H_1^f]$ is the event that the coin is fair and the first toss resulted heads, $[H_1^b]$ is the event that the coin is biassed and the first toss resulted heads, and similarly for $[T_1^f]$ and $[T_1^b]$. Clearly,

$$[H_1^f] := \left\{ (x, y_1, y_2, y_3) \in W \;\middle|\; x = b_1,\ y_1 = h_1 \right\}$$

and

$$[H_1^b] = \left\{ (w, x, y_1, y_2, y_3) \in W \;\middle|\; w = a,\ y_1 = h_1 \right\}$$

and similarly for $[T_1^f]$ and $[T_1^b]$. In other words, the outcome of the first coin toss as well as fairness and biassedness of the coin can be distinguished in Σ_{env}, but not whether the coin was forged to be second or third type as each element in Σ_{env} contains both versions or neither.

Since we consider the chance after the first coin toss,

$$\tau(w) = \begin{cases} 2 \text{ on } W_1 \\ 3 \text{ on } W_2 \end{cases}$$

and the atoms of Σ_τ are those sets of quadruples in W_1 or quintuples in W_2 for which the first two or three elements are fixed while the last two both range through heads and tails. Clearly,

$$\Sigma_{\text{env}} \neq \Sigma_\tau.$$

If we want to consider the chances of the various biasses as well as the chances of the outcomes of the second coin toss, we set

$$\Sigma_{\text{chancy}} := \overline{\{\ \emptyset\ ,\ [B_1]\ ,\ [B_2]\ ,\ [B_3]\ ,\ [H_2]\ ,\ [T_2]\ ,\ W\ \}}$$

In order to make the computations easier, let us modify the biasses for $[B_2]$ and $[B_3]$ from the previous examples. Suppose that the coin of the second kind has a chance to land heads by $0.4 = 2/5$, and the coin of the third kind has a chance to land heads by $0.75 = 3/4$. Otherwise the manufacturing process is the same as in Section 3.2.2.

For any $w \in W$ and $A \in \Sigma_{\text{chancy}}$, we have to compute $ch(w, A)$. As this in its first variable has to be measurable with respect to Σ_{env}, it is constant on the atoms of Σ_{env}, namely, on $[H_1^f]$, $[T_1^f]$, $[H_1^b]$, $[T_1^b]$. What is, for example $ch(w, [B_2])$ when $w \in [H_1^f]$? In this case w is a possible world on which the coin is fair. So it could not be manufactured the second way, and $ch(w, [B_2]) = 0$. Similarly $ch(w, [B_2]) = 0$ when $w \in [T_1^f]$. What is $ch(w, [B_2])$ when $w \in [H_1^b]$? Given the coin was manufactured to be biassed, the chance that it lands heads is $1/2 \cdot 2/5 + 1/2 \cdot 3/4 = 1/5 + 3/8 = 23/40$. Given the coin was manufactured to be biassed, the chance that it is manufactured in the second way and that it lands heads $1/2 \cdot 2/5 = 1/5$. So, $ch(w, [B_2]) = (1/5)/(23/40) = 8/23$ when $w \in [H_1^f]$. Similarly, we have $ch(w, [B_2]) = (1/2 \cdot 3/5)/(1/2 \cdot 3/5 + 1/2 \cdot 1/4) = (3/10)/(17/40) = 12/17$ when $w \in [T_1^b]$. So we have

$$ch(w, [B_2]) = \begin{cases} 0 \text{ if } w \in [H_1^f] \cup [T_1^f] = W_1 \\ 8/23 \text{ if } w \in [H_1^b] \\ 12/17 \text{ if } w \in [T_1^b] \end{cases}$$

Similarly the values for B_3 can be computed. Moreover,

$$ch(w,[H_2]) = \begin{cases} 0.5 \text{ if } w \in [H_1^f] \cup [T_1^f] = W_1 \\ \alpha \text{ if } w \in [H_1^b] \\ \beta \text{ if } w \in [T_1^b] \end{cases}$$

Where

$$\alpha = (1/2 \cdot 2/5 \cdot 2/5 + 1/2 \cdot 3/4 \cdot 3/4)/(2/5 \cdot 1/2 + 3/4 \cdot 1/2) \quad (1)$$

and

$$\beta = (1/2 \cdot 3/5 \cdot 3/5 + 1/2 \cdot 1/4 \cdot 1/4)/(3/5 \cdot 1/2 + 1/4 \cdot 1/2)$$

Similarly for T_2.

Consequently, it is easy to construct examples when objective chance relative to some event algebra Σ_{env} is considered with Σ_{env} different from Σ_τ for any stopping time τ. The notion of time may in fact be entirely missing from the model, objective chance can still be made sense. So for the objective chance it is not Σ_τ that is important, but Σ_{env} expressing the events relative to which objective chance is considered. Σ_{env} may or may not agree with some Σ_τ.

To make this point even stronger, consider the situation when we can allow various space-time models. In other words, some of the possible worlds correspond to certain space-time models, others correspond to other space-models. Then we can ask, given a model, what is the objective chance that something happens in that model. Clearly, this is not a chance associated to a certain time, but just to various space-time models. In such a case Σ_{env} would contain sets of space-time models.

3.4 Impossible Events Have Zero Chance

There is a very important property of objective chances that we have not talked about yet, and that is entirely missed by both Lewis's original paper as well as by later literature on the Principal Principle. Namely, consider an $E \in \Sigma_{\text{env}}$ and an $A \in \Sigma_{\text{chancy}}$ such that $E \cap A = \emptyset$. In other words, if we fix the chancy setup to satisfy the properties of E, then there is no way for event A to come about because none of the possible worlds in E are also in A. Therefore an objective chance function must not give non-zero chance of A for possible worlds in E. But assuming only that *ch* is measurable with respect to Σ_{env} in its first variable and that it is a probability distribution over Σ_{chancy} in its second variable, *ch* might still violate this condition. So we have to additionally require it for an objective chance. Consider the following specific example.

Let us go back now to the original example, but suppose we add a single further possible world to it:

$$W := \left\{ (x,y) \;\middle|\; (x = b_1 \vee x = b_2 \vee x = b_3) \wedge (y = h \vee y = t) \right\} \cup \{(b_4, h)\}$$

In other words, with b_4 bias, the coin only lands on heads. For example, both sides of the coin is heads. With this modification,

$$\Sigma_{\text{env}} = \overline{\{\emptyset, [B_1], [B_2], [B_3], [B_4]\}}$$

with $[B_4] = \{(b_4, h)\}$, and

$$[T] = \{(b_1, t), (b_2, t), (b_3, t)\}, \qquad [H] = \{(b_1, h), (b_2, h), (b_3, h), (b_4, h)\}$$

Clearly, $[B_4] \cap [T] = \emptyset$. That is no surprise, because in our setup, if the coin has the b_4 bias, it cannot land heads. So is it possible that $ch((b_4, h), [T]) \neq 0$? No it is not. The atom $[B_4]$ of Σ_{env} contains no elements in $[T]$, meaning that on $[B_4]$, the outcome of the chancy event can never be in $[T]$. Since this is an impossible event in $[B_4]$, it has no chance to occur in $[B_4]$, and hence the chance of $[T]$ in $[B_4]$ must be 0.

This can of course be told in general: if $E \in \Sigma_{\text{env}}$ and $A \in \Sigma_{\text{chancy}}$ is such that $E \cap A = \emptyset$, then the objective chance $ch(\cdot, A)$ must be 0 on E as there is no way A will occur on E. This is an additional requirement on objective chance.

3.5 Frequentist's Objective Chance

In this final example, we consider the frequentist's model of objective chance, and argue that it can also be described with two subalgebras, Σ_{env} and Σ_{chancy} as the other cases. Consider the situation when the set of possible worlds contains only 8 possible worlds: 3 independent tosses of a fixed coin, each with two possible outcomes, heads and tails. This set of possible worlds has the following elements:

$$W = \{(h, h, h), (h, h, t), (h, t, h), (t, h, h), (t, t, h), (t, h, t), (h, t, t), (t, t, t)\}$$

The event that the i'th coin toss is heads is

$$[H_i] = \{(y_1, y_2, y_3) \in W | \pi_i((y_1, y_2, y_3)) = h\}$$

that it is tails is

$$[T_i] = \{(y_1, y_2, y_3) \in W | \pi_i((y_1, y_2, y_3)) = t\}$$

We can take $\Sigma_{\text{chancy}} := 2^W$, as all events are chancy.

Now, a frequentist would say that the chance of heads, that is, of any of $[H_i]$ at time 0 is 1 on the possible world (h, h, h), is $2/3$ on (h, h, t), (h, t, h), and (t, h, h), it is $1/3$ on (t, h, h), (t, t, h), (t, h, t), and 0 on (t, t, t). Accordingly, since in this model the chance is determined by the frequencies of h and t on the possible worlds, Σ_{env} has to be set to

$$\Sigma_{\text{env}} := \overline{\{\{(h, h, h)\}, \{(h, h, t), (h, t, h), (t, h, h)\}, \{(t, t, h), (t, h, t), (h, t, t)\}, \{(t, t, t)\}\}}$$

and $ch_0((h, h, h), [H_i]) = 1$, $ch_0(w, [H_i]) = 2/3$ if $w \in \{(h, h, t), (h, t, h), (t, h, h)\}$, $ch_0(w, [H_i]) = 1/3$ if $w \in \{(t, t, h), (t, h, t), (h, t, t)\}$, $ch_0((t, t, t), [H_i]) = 0$. And clearly,

$ch_0(w, [T_i]) = 1 - ch_0(w, [H_i])$. Now, as we set $\Sigma_{\text{chancy}} = 2^W$, we have to define $ch_0(w, S)$ for any subset of W. What is for example, $ch_0(w, [H_1] \cap [H_2])$? The first idea would be $ch_0(w, [H_1] \cap [H_2]) = ch_0(w, [H_1]) \cdot ch_0(w, [H_2])$ as the tosses are independent. However, we immediately realize, that as in this model, we assumed that the coin is tossed only 3 times, the tosses are not independent any more once we fix the chance.[3] For example, if the chance of heads is assumed to be $2/3$, and the first two tosses were both heads, then we know the third must be tails. The objection could be that before the third toss, we do not know that there will be no more tosses. However, that means that our model of the world is wrong, we have to allow possible worlds with any long sequences of tosses, and in that case, the longer the sequence is, the closer the chance will get to result independence between the 2nd and the 3rd coin toss. But given the frequentist view, they will only be perfectly independent if we assume infinitely long sequences.

So what is $ch_0(w, [H_1] \cap [H_2])$? Clearly, it is $ch_0(w, [H_1] \cap [H_2]) = ch_0(w, [H_2] | [H_1]) \cdot ch_0(w, [H_1])$. But what is $ch_0(w, [H_2] | [H_1])$? This can again be obtained by the frequentist view. For example, if $w \in \{(h, h, t), (h, t, h), (t, h, h)\}$, then given that the result of the first toss is h, the rest of the tosses are either ht or th, and hence $ch_0(w, [H_2] | [H_1]) = 1/2$. then for this w, $ch_0(w, [H_1] \cap [H_2]) = 1/3$, and not $4/9$. Similarly, $ch_0(w, [H_1] \cap [H_2] \cap [H_3]) = 0$, and so on.

It is easy to see that this frequentist objective chance does not correspond to any Σ_τ either, because chance depends on the whole history, but Σ_{env} is smaller than the full $\Sigma_4 = \mathcal{P}(W)$.

4 The Subjectivist Agent's Model of Objective Chance

According to the forgoing, I propose the following definition of a model of possible worlds with objective chance:

Definition 4.1 (Possible World Model with Objective Chance) $(W, \Sigma, \Sigma_{\text{env}}, \Sigma_{\text{chancy}}, ch)$ *is a possible world model with objective chance if*

- W *is a set (of possible worlds)*
- Σ *is a* σ*-algebra on* W
- Σ_{env} *is a* σ*-subalebra of* Σ
- Σ_{chancy} *is a* σ*-subalebra of* Σ
- $ch : W \times \Sigma_{\text{chancy}} \to [0, 1]$ *such that*
  - *For all* $w \in W$, $A \mapsto ch(w, A)$ *is a probability measure on* Σ_{chancy}
  - *For all* $A \in \Sigma_{\text{chancy}}$, $w \mapsto ch(w, A)$ *is a* Σ_{env}*-measurable function on* W
  - *If for some* $E \in \Sigma_{\text{env}}$ *and* $A \in \Sigma_{\text{chancy}}$, $E \cap A = \emptyset$, *then* $ch(w, A) = 0$ *for all* $w \in E$.

[3]Hence we strongly disagree with the treatments of some other authors such as Pettigrew (1986) on page 6.

Of course, if for each time t we have some Σ_{env} and some *ch*, then we can index them with time, but time is not essential for the definition of objective chance.

This definition works for infinite W as well. A sigma algebra Σ is closed under countably infinite union and complementation, and contains W.

Again, as in Section 3.1.4, for each w, the chance of any event in $\overline{\Sigma_{\text{env}} \cup \Sigma_{\text{chancy}}}$ can be defined with respect to Σ_{env}. Given $(W, \Sigma, \Sigma_{\text{env}}, \Sigma_{\text{chancy}}, ch)$, for each $w \in W$, there is a *unique* extension of $ch(w, \cdot)$ to $\overline{\Sigma_{\text{env}} \cup \Sigma_{\text{chancy}}}$ such that for the extended chance, which we shall denote by $\overline{ch}$ also satisfies the above requirements, and $(W, \Sigma, \Sigma_{\text{env}}, \overline{\Sigma_{\text{env}} \cup \Sigma_{\text{chancy}}}, \overline{ch})$ is a possible world model with objective chance: For all $w \in W$, $E \in \Sigma_{\text{env}}$, set

$$\overline{ch}(w, E) := \begin{cases} 1 \text{ if } w \in E \\ 0 \text{ if } w \notin E \end{cases}$$

Because of the last requirement in the definition of a possible world model with objective chance, there is no other way to define $\overline{ch}$ on Σ_{env}. If $w \notin E \in \Sigma_{\text{env}}$, then it is in $E^{\perp} \in \Sigma_{\text{env}}$, and since $E^{\perp} \cap E = \emptyset$, by the last requirement $\overline{ch}(w, E) = 0$ for all $w \in E^{\perp}$ as long as $\overline{ch}$ is an objective chance function. Then for any $A \in \Sigma_{\text{chancy}}$, $E \in \Sigma_{\text{env}}$, and $w \in E$,

$$\overline{ch}(w, E \cap A) = \overline{ch}(w, E) \cdot \overline{ch}(w, A|E)$$

must hold if $\overline{ch}(w, \cdot)$ is to be a probability distribution. As $w \in E$, $\overline{ch}(w, E) = 1$, and $\overline{ch}(w, A|E) = \overline{ch}(w, A)$. But as $\overline{ch}$ is an extension of *ch*, $\overline{ch}(w, A) = ch(w, A)$. Furthermore, when $w \notin E$, $\overline{ch}(w, E \cap A) = 0$. Hence I have shown that there is only one way of defining $\overline{ch}$, namely with the formula

$$\overline{ch}(w, E \cap A) = \overline{ch}(w, E) \cdot ch(w, A)$$

As all elements in $\overline{\Sigma_{\text{env}} \cup \Sigma_{\text{chancy}}}$ can be written as the union of sets of the form $E \cap A$ as above (the event algebra Σ is generated by Σ_{env} and Σ_{chancy}), this defines a unique probability measure $S \mapsto \overline{ch}(w, S)$ on $\overline{\Sigma_{\text{env}} \cup \Sigma_{\text{chancy}}}$.

The above rigorous definition of objective chance opens the possibility of investigating the Principal Principle and admissible evidence in a mathematically precise way, which I shall do elsewhere.

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